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Approximate inverse spectral transform for the Korteweg–de Vries equation

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Abstract. The numerical solution of the inverse spectral transform for the Korteweg-de Vries equation is rather time consuming and can give rise to instabilities. Motivated by such drawbacks, the authors propose a formalism which allows one to derive in closed form the solution of the Cauchy problem for the Korteweg-de Vries equation for short times. Such a solution is obtained by going over to a Padé approximant of the time-evolved reflection coefficient.

1. Introduction

As is well known, for linear partial differential equations (PDEs) of the evolution type, $u_t = F(u, u_{xx}, u_{xx}, \ldots)$, and for rapidly decreasing initial data, the solution of the Cauchy problem can be achieved through the (direct and inverse) Fourier transform (FT). Moreover, it is now established that the Cauchy problem can also be solved for a large class of nonlinear PDEs (NPDEs) (Gardner *et al* 1967, Zakharov and Shabat 1971, Ablowitz *et al* 1974, Calogero and Degasperis 1982) through a nonlinear extension of the FT, namely the so-called (direct and inverse) spectral transform (DST and IST).

Let us consider, for instance, the KdV equation:

$$u_t = u_{xxx} - 6uu_x. \tag{1}$$

One can associate with it the spectral problem (SP)

$$-\psi_{xx} + u\psi = k^2\psi \tag{2}$$

which, given u belonging to L_2^1 (Deift and Trubowitz 1979), allows us to define uniquely the spectral data:

$$S[u] = \{R(k),\ k \in \mathbb{R};\ p_j,\ \rho_j\ (j = 1, 2, \ldots, N)\} \tag{3}$$

where $R(k)$ is the reflection coefficient, $-p_j^2$ are the discrete eigenvalues and ρ_j are the corresponding wavefunction normalisation coefficients. For the explicit definition of these quantities we refer to Calogero and Degasperis (1982, p 18). Starting from an initial datum $u_0(x)$, the DST (2) gives the spectral data $S_0 = S[u_0]$ which, if u_0 evolves according to the NPDE (1), evolves according to a linear ordinary differential equation (LODE), which can be explicitly integrated. Finally, the solution of the Cauchy problem can be accomplished through the IST which, in this case, consists essentially of a linear integral equation of Fredholm type.

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The procedure described above can be summarised in the following scheme:

$$\begin{array}{ccc} u(x,0)=u_0(x) & \xrightarrow{\text{DST}} & S[u_0] \\ \downarrow \text{NPDE} & & \downarrow \text{LODE} \\ u(x,t) & \xleftarrow{\text{IST}} & S[u]. \end{array}$$

However, while in the DST, apart from a large class of exact solutions (see, for instance, Calogero and Degasperis (1982, pp 418-44)), one has at his disposal an effective and fast numerical algorithm (Osborne 1984), the solution of the IST is in general more cumbersome, the numerical algorithm is slow and can give rise to instabilities. Such difficulties can be by-passed if one is just interested in the asymptotic (i.e. 'large t and x') behaviour of the solution (Ablowitz and Segur 1977).

In many instances, however, the knowledge of the solution for small t could be relevant: this can be accomplished by going over to the Padé approximant series of the evolved spectral data (Baker 1981). In fact, in this case, the reflection coefficient is a rational function of the spectral variable and thus the IST can be completely solved (Sabatier 1984). To do so, Sabatier (1984), following Calogero and Degasperis (1982), had to reconstruct the potential separately for $x<0$ and $x>0$.

In the following we shall derive a procedure such that this splitting is not necessary, at the expense of the formal introduction of a δ function in the origin, which for reflection coefficients sufficiently well behaved, disappears. Closely related results have been derived, with a different aim in mind, by Pechenick and Cohen (1981), and Moses and Prosser (1984).

2. Rational reflection coefficients

As is well known (Marchenko 1968), the IST for the SP (2) is functionally stable, so that a good Padé approximant of a given spectrum (3) provides a good approximation for the corresponding potential.

The Padé approximant of a given reflection coefficient $R(k)$ obtains, at least in its simplest version, from the Taylor expansion around the origin:

$$R(k)=\sum_{l=0}^{\infty} c_l k^l \simeq P_m(k)/Q_n(k) \equiv R_p(k) \tag{4}$$

where

$$P_m(k)=-1+\sum_{l=1}^{m} a_l k^l \qquad Q_n(k)=1+\sum_{l=1}^{n} b_l k^l.$$

We notice that the structure of P_m and Q_n guarantees the condition $R_p(0)=-1$.

It is easy to see that the reality of the potential, expressed in terms of the reflection coefficient by the property:

$$R(k)=R^*(-k^*)$$

which implies in (4) $c_l^*=(-1)^l c_l$, is preserved by the Padé approximant $R_p(k)$ and implies

$$b_l=(-1)^l b_l^* \qquad a_l=(-1)^l a_l^*. \tag{5}$$

Moreover $R(k)$, being obtained from a well behaved potential $u_0(x)$, must vanish asymptotically in k: this implies that, in any $R_p(k)$, we must have $n > m$. Finally, the property (5) easily implies† that the zeros of P_m and Q_n are either purely imaginary or symmetric with respect to the imaginary k axis.

As is well known (Neugebauer and Meinel 1983) the reconstruction of the potential from the spectral data can be carried out in two steps: first, one obtains the potential and the wavefunction corresponding to the continuous part of the spectrum (in our case, corresponding to the poles of $R_p(k)$ lying in the lower half of the complex k plane) and then by successive Bäcklund transformations (BTS) one dresses it up with poles in the upper half of the complex k plane (which may or may not correspond to bound states). Thus it is appropriate to write $R_p(k)$ in the following form:

$$R_p(k) = R_c(k) R_b(k) \tag{6}$$

where $R_c(k)$ has poles in the lower half plane and $R_b(k)$ has poles in the upper half plane. Assuming $Q_n(k)$ has s zeros in the upper half plane, so that $Q_n(k) = Q_s^{(+)}(k) Q_{n-s}^{(-)}(k)$, where $Q_r^{(\pm)}(k)$ has r zeros in the upper (lower) half plane, we have

$$R_b(k) = Q_s^{(+)}(-k)/Q_s^{(+)}(k) \qquad R_c(k) = P_m(k)/(Q_s^{(+)}(-k) Q_{n-s}^{(-)}(k)). \tag{7}$$

In order to obtain the explicit form of the potential on the whole x axis corresponding to $R_c(x)$, we write it down as a sum of (simple) partial fractions:

$$R_c(k) = -\mathrm{i} \sum_{j=1}^{n} \frac{d_j g_j}{k + \mathrm{i} g_j} \qquad \left(\operatorname{Re} g_j > 0; \sum_{j=1}^{n} d_j = 1 \right) \tag{8}$$

d_j, g_j being definite functions of the coefficients c_l appearing in equation (4).

Starting from (8) one can evaluate the kernel $M(x)$ of the Gel'fand-Levitan-Marchenko equation:

$$K_c(x, y) + M_c(x + y) + \int_x^{\infty} \mathrm{d}z \, K_c(x, z) M_c(z + y) = 0 \qquad (y \geqslant x)$$

whose solution is

$$K_c(x, y) = \theta(-x)\theta(x^2 - y^2) \left(A(x) + \sum_{k=1}^{n-1} D_k(x) [\exp(-\omega_k y) - R_c(-\mathrm{i}\omega_k) \exp(\omega_k y)] \right)$$

where ω_k $(k = 1, \ldots, n-1)$ are the strictly positive roots of the equation

$$\sum_{j,k=1}^{n} d_j d_k \frac{\omega + g_j - g_k}{(\omega + g_j)(\omega - g_k)} = 0$$

and the functions $A(x)$, $D_k(x)$ are given as ratios of determinants by the formulae:

$$A = \det \Delta^{(1)} / \det \Delta \qquad D_k = \det \Delta^{(k+1)} / \det \Delta \tag{9}$$

where the matrices Δ, $\Delta^{(l)}$ are given by

$$\Delta_{rs} = (g_r)^{-1} \delta_{s1} + \alpha_{rs}(x)(1 - \delta_{s1})$$

$$\Delta_{rs}^{(l)} = \Delta_{rs}(1 - \delta_{ls}) + 1 \cdot \delta_{ls}$$

$$\alpha_{rs}(x) = (g_r - \omega_s)^{-1} \exp(-\omega_s x) - R_c(\mathrm{i}\omega_s)(g_r + \omega_s)^{-1} \exp(\omega_s x).$$

† This is proved just by setting $k = \mathrm{i}y$ and noting that $P_m(y)$ and $Q_n(y)$ have real coefficients and thus real or complex conjugate zeros.

Consequently†:

$$u_c(x) = -2\delta(x)\left(A(0) + \sum_{k=1}^{n} D_k(0)(1 - R_c(-i\omega_k))\right)$$
$$-2\theta(-x)\left(\tilde{A}(x) + \sum_{k=1}^{n} \{\tilde{D}_k(x)[\exp(-\omega_k x) - R_c(-i\omega_k)\exp(\omega_k x)]\right.$$
$$\left. - \omega_k D_k(x)[\exp(-\omega_k x) + R_c(-i\omega_k)\exp(\omega_k x)]\}\right)$$

where

$$\tilde{A} = \det \tilde{\Delta}^{(1)}/\det \Delta \qquad \tilde{D}^{(k)} = \det \tilde{\Delta}^{(k+1)}/\det \Delta \tag{10}$$
$$\tilde{\Delta}^{(l)}_{rs} = \Delta_{rs}(1 - \delta_{ls}) + \beta_s(x)\delta_{ls}$$

$$\beta_s(x) = \sum_{k=1}^{n} \omega_k D_k(x)[(g_s - \omega_k)^{-1}\exp(-\omega_k x) + R_c(i\omega_k)(g_s + \omega_k)^{-1}\exp(\omega_k x)].$$

We recall now that, according to the results of Neugebauer and Meinel (1984), the potential corresponding to the reflection coefficient $R_p(k)$ (6) is given by

$$u_p(x) = (-1)^s[u_c(x) - 2i \det \tilde{V}/\det V]$$

where s is the number of poles of $R_p(k)$ in the upper half of the complex k plane and the $2s \times 2s$ matrices $\tilde{V}$ and V are expressed by the formulae:

$$V_{p1} = 1 \qquad V_{p2} = (\alpha_p)^{-1} \qquad V_{p,q+2} = k_p V_{p,q}$$
$$\tilde{V}_{p,q} = V_{p,q}(1 - \delta_{2s,q}) + (k_p)^s \delta_{q,2s}.$$

The functions $\alpha_p(x)$ are the so-called intermediate wavefunctions, depending only on the solution of the Schrödinger equation (2) corresponding to the potential $u = u_c(x)$ and to $k^2 = k_p^2$ where k_p ($p = 1, \ldots, s$) are the zeros of $Q_s^{(+)}(k)$ which, as we noticed before, are located in the upper complex k plane and are either purely imaginary or symmetric with respect to the imaginary axis. The intermediate wavefunction is given by

$$\alpha_p(x) = (d/dx)[\ln(\eta_p \psi^{(+)}(x, k_p) + \psi^{(+)}(x, -k_p))] - ik_p \tag{11}$$

with

$$\psi^{(+)}(x, k_p) = \exp(ik_p x) - 2\theta(-x)\left[A(x)\frac{\sin k_p x}{k_p} + \sum_{l=1}^{n-1} D_l(x)\right.$$
$$\left. \times\left(\frac{\sin(k_p + i\omega_l)x}{k_p + i\omega_l} - R_c(-i\omega_l)\frac{\sin(k_p - i\omega_l)x}{k_p - i\omega_l}\right)\right]. \tag{12}$$

It is worthwhile noticing that equation (11) can be expressed completely in algebraic form through the function $A(x)$, $D_k(x)$ (equation (9)) and $A(x)$, $D_k(x)$ (equation (10)), and moreover that, even if formula (12) contains a θ function, equation (11) does not contain any δ function in the origin.

† One can check that, if $R_c(k)$ decays faster than k^{-1} as $|k| \to \infty$, the coefficient of the δ function vanishes. This same result has been obtained independently by Pechenick and Cohen (1981) and Moses and Prosser (1984).

The coefficients η_p appearing in (11) are related to the normalisation coefficients ρ_p (3) of the eigenfunctions $\psi^{(+)}(x, k_p)$ and to the value of the reflection coefficient $R_c(k)$ at the points k_p when, as is the case here, the reflection coefficient is analytic, except for a finite number of poles, in the whole complex k plane. Thus, we have

$$\eta_p = -(i\rho_p/2k_p) + R_c(k_p) \tag{13}$$

and $\rho_p = 2ik_pR_c(k_p)$ if k_p corresponds to a bound state and otherwise $\rho_p = 0$. Condition (13) thus implies $\eta_p = 0$ whenever k_p corresponds to a bound state†.

For completeness and for further use it is worthwhile noticing that the condition $\eta_p = 0$ (in agreement with Calogero and Degasperis (1982)) also holds whenever $\operatorname{Im} k_p < 0$.

3. Time evolution via Padé approximants

It is well known (Gardner *et al* 1967, Calogero and Degasperis 1982) that, when the potential u of the Schrödinger equation (2) evolves according to the KdV equation, the corresponding evolution of the spectral data is given by

$$\begin{aligned} R(k, t) &= R(k, 0)\exp(-8ik^3t) \\ p_j(t) &= p_j(0) \\ \rho_j(t) &= \rho_j(0)\exp(-8p^3t). \end{aligned} \tag{14}$$

Hence, of course, a rational dependence on k of the reflection coefficient is not preserved by the time evolution; this drawback, however, can easily be overcome, at least for small times, as pointed out in the introduction, by replacing $R(k, t)$ by its Padé approximant. The simplest Padé approximant to equation (14) is easily seen to be

$$R_p(k, t) = R(k, 0)\frac{1-4ik^3t}{1+4ik^3t} \equiv R(k, 0)R_e(k, t) \tag{15}$$

the difference $|R_p - R|$ being $O(t^2)$.

By taking into account the integral Wronskian relations connecting the time evolution of the potential (Calogero and Degasperis 1982, p 90) we easily derive the NPDE for u which has the exact time evolution of the reflection coefficient given by (15):

$$(1-\tfrac{1}{4}t^2\boldsymbol{L}^3)u_t = \boldsymbol{L}u_x = u_{xxx} - 6uu_x. \tag{16}$$

Formula (16) clearly shows that, for small times, the NPDE (16) approximates the KdV equation (1).

We notice that the factor $R_e(k, t)$ can be factorised as

$$R_e(k, t) = \prod_{j=1}^{3} \frac{k-k_j(t)}{k+k_j(t)} \tag{17}$$

where two of the k_j are symmetric with respect to the imaginary axis in the upper half complex k plane while the third is purely imaginary in the lower half complex k plane,

† Equation (13) corrects the statement by Calogero and Degasperis (1982, pp 378–85).

in such a way that

$$\sum_{j=1}^{3} (k_j)^n = 0 \qquad (n = 1, 2).$$

Due to the structure (17) $R_e(k)$ can be treated in the same manner as $R_b(k)$ (7) even if it also has poles in the lower half plane†. Thus $R_e(k)$ can be included in $R_b(k)$ by adding three time-dependent poles to the existing s poles in the 'zero time' Padé approximant.

4. Conclusions

Of course, the type of Padé approximant we have used is suitable for expressing the behaviour of the reflection coefficient for small k, and thus the behaviour of the potential for large x. For a more detailed reconstruction of the potential one would also need the Padé approximant representation of $R(k)$ for large k, and then the appropriate matching between the two regions. Naturally both Padé approximants can be handled by the technique introduced here. In such a way one can think of describing the small-time behaviour of the solution of the KdV equation for a *generic* initial datum.

Through this technique one can also try to clarify the small-time behaviour of the background radiation solutions of some 'pathological' NPDE with x dependent coefficients (Calogero and Degasperis 1982, pp 234–58). Work is in progress in both directions.

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† It is perhaps worthwhile to notice that, since the time evolution (14) is just a phase factor, the proper Padé approximants should have $n = m$ at any order, always implying the structure (17).